Eur. Phys. J. B **56**, 53–63 (2007)
DOI: 10.1140/epjb/e2007-00091-9

THE EUROPEAN PHYSICAL JOURNAL B

Evolution of imitation networks in Minority Game model

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Received 20 December 2006
Published online 4 April 2007 –

Abstract. The Minority Game is adapted to study the "imitation dilemma", i.e. the tradeoff between local benefit and global harm coming from imitation. The agents are placed on a substrate network and are allowed to imitate more successful neighbours. Imitation domains, which are oriented trees, are formed. We investigate size distribution of the domains and in-degree distribution within the trees. We use four types of substrate: one-dimensional chain; Erdös-Rényi graph; Barabási-Albert scale-free graph; Barabási-Albert 'model A' graph. The behaviour of some features of the imitation network strongly depend on the information cost ϵ, which is the percentage of gain the imitators must pay to the imitated. Generally, the system tends to form a few domains of equal size. However, positive ϵ makes the system stay in a long-lasting metastable state with complex structure. The in-degree distribution is found to follow a power law in two cases of those studied: for Erdös-Rényi substrate for any ϵ and for Barabási-Albert scale-free substrate for large enough ϵ. A brief comparison with empirical data is provided.

PACS. 89.65.-s Social and economic systems – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.-r Probability theory, stochastic processes, and statistics

1 Introduction

The Minority Game (MG) [1,2] emerged in the last decade as a prototype model of socio-economic behaviour and some call it (perhaps exaggerating) the "Ising model of econophysics". To formulate it shortly, an odd number of agents, each having two strategies, chooses among two options. Those who managed to choose the option which turned to be the minority, win a point, the others loose a point. A record of winning options is kept and each agent can see the outcomes in last M rounds. The strategies map the M-tuples of outcomes into suggested choices. Each strategy has a virtual score, indicating how often it would suggest the winning choice, provided the strategy was used by the agent. The actually used strategy is the one with the highest score.

Such simple dynamical system (in fact it is not a game in game-theoretic sense, because it is purely deterministic) has extremely rich behaviour with a phase transition [3,4], which attracted much attention [5–14] and several analytic approaches [15–25] were tried to elucidate its properties. Despite of all these efforts, the exact solution of MG in its original formulation [2] eludes us.

Besides the principal interest in dealing with a challenging model [26,27], MG is frequently used as a testing bed or framework for various econo- and sociophysical applications [27,28]. These frequently imply modifications of MG in specific directions. Among them, the grand-canonical MG as a model for stock market fluctuations deserves special attention [29–34].

However, in this work we focus on another offshoot of MG. In classical formulation, the agents do not interact directly, but exchange information only through the global authority, which is the record of winning choices. Several ways were proposed how the agents may interact locally [35–40].

Some time ago we introduced a model [41,42], in which the local interaction of agents in Minority Game occurs by imitation. We used the simplest possible social structure. Agents placed on a linear can look at their left neighbours. If the neighbour of an agent is more successful, the agent follows the actions of the neighbour. Compact imitation domains performing the same action emerge. Originally we asked, whether the imitation may bring the assembly of agents closer to the optimal state by lowering the volatility. Partially it did happen, and in the crowded phase

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of MG, the imitation domains were shown to lead to enhanced performance, i.e. decrease of the volatility [41]. On the other hand, the individual profit from imitation overweighted the respect to global benefits and the domains did not stop growing when the optimum was reached, thus turning the performance into the worse again.

This is one of the instances of the effect we call "imitation dilemma". The simplest way to formulate it is the following. If the agents have limited capacity to search for useful resources, cheating is an advantage, because by copying from other agents who either do their own search or copy in their turn, the subset of all resources searched by the imitator is effectively enlarged. On the other hand, this cannot be a universal strategy, for this would mean that everybody is copying and nothing is really searched. Absence of a strategy good for all is the situation which generically happens also in MG, therefore we consider MG as a good framework to study the effects of imitation in general.

Imitation in MG was subsequently elaborated by other groups [43–47]. New surge of interest in this type of problems came in the wake of the work by Anghel et al. [48]. While our study [41,42] was limited on the simplest one-dimensional directed topology of the interactions between agents, in reference [48] the agents resided at the vertices of a random graph, specifically on Erdös-Rényi (ER) graph [49] where an edge between any pair of vertices is present with probability q, independently of the presence or not of other edges. Upon this substrate graph the agents form an imitation network. Every agent adopts the strategy of their best performing neighbour, unless she is herself better than any if her neighbours. Contrary to our approach, the imitation does not propagate further, so that large domains acting in unison are not formed and the feedback on the MG dynamics is therefore very weak, if any. Nevertheless, the improvement of performance in the crowded phase was found here too.

The main result was that the imitation network has power-law degree distribution. The value of the exponent was $\simeq$1 and was remarkably universal, showing no dependence on either the MG parameters N and M, or the edge probability q.

This work was followed by careful numerical and analytical scrutiny [50–54]. The question appeared naturally, what happens if we use other networks as a substrate, instead of ER graph. It was found [55,56] that the behaviour is strongly influenced by the type of the substrate. Three types of the latter can be distinguished, namely the ER graph, 1D regular lattice and a scale-free network, be it the Barabási-Albert (BA) graph [57] or another example from the same category. Roughly speaking, the efficiency was worst in 1D case, best in ER case, and scale-free networks were somewhere in between. What concerns the structure of the imitation network, power-law degree distribution was reported on both ER and scale-free graphs. The exponent was 1 in the former, but significantly larger in the latter case. However, to be frank, our reading of reference [56] suggests that for BA graph there may be important influence of the memory length M. The data show clear power law only for $M = 2$, but deviate from it the more the larger M is. Contrary to the ER case, for scale-free substrate the scale-free structure of the imitation network does not seem universal.

In this work we ask similar questions for our original model [41]. However, we shall broaden our focus to features which are typical to our approach, especially the structure of imitation domains and influence of the cost of information, i.e. the fee the imitators must pay to the imitated.

We should also stress that the study of imitation networks is not a purely academic business. Recent empirical studies in this direction exist [58], based on the study of delayed correlation functions of various stocks on the market.

2 Minority Game with imitation

Let us describe the structure of our model [41] generalised to arbitrary substrate graph. There are N agents with memory M playing standard Minority Game, with options ±1. Each agent has two strategies distinguished by index $\sigma = \pm1$. If the M-tuple of last winning options is $\mu \in \{-1,+1\}^M$, the strategy σ belonging to agent i suggests the action $a^{\mu}_{i,\sigma}$. The strategies are given scores $U_{i,\sigma}$ according to standard minority rule. The best strategy of the agent i is $S(i) = \mathrm{sign}(U_{i,+1} - U_{i,-1})$. However, the action taken by an agent is not just the action suggested by her best strategy, but it is influenced by the social network the agents are placed on.

The network is represented by a graph $(\mathcal{V}, \mathcal{E})$, where each of the vertices from the vertex set $\mathcal{V} = \{1, 2, \ldots, N\}$ hosts one agent and the edges from $\mathcal{E}$ specify who can share information wit whom. In [41] we considered directed edges, but here the edges will be always undirected. Let us denote $\Gamma_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\} \subset \mathcal{V}$ the local neighbourhood of the agent i. When the agent tries to figure out what action to take, she first compares her own wealth with the wealth of agents in her neighbourhood. If she happens to be the wealthiest, she is classified as a leader, otherwise she is a follower. Denote $W_i(t)$ the wealth of the agent i at time t. The status of the agent i is described by the variable

$$l_i = \theta(\max_{j\in\Gamma_i} W_j - W_i) \tag{1}$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ otherwise. If the agent is a follower, $l_i = 1$, then we denote $L(i) = \arg\max_{j\in\Gamma_i} W_j$ the agent imitated by i, or her local leader. The action actually taken by the agent i is then given by the recurrent formula

$$a_i = l_i\, a_{L(i)} + (1 - l_i)\, a^{\mu}_{i,S(i)}. \tag{2}$$

In plain words, the leaders use their best strategies, while the followers copy the action from their local leaders, which, in their turn, can also be followers, thus we must continue along the imitation chain until we end at a leader. Here we differ substantially from the model by Anghel et al.

As we have already said, the strategies' scores evolve according to the usual minority rule. On the other hand, for the increase of the wealth of an agent we adopt slightly more complicated prescription. Indeed, one of the most important ingredients of our model is the assumption that the information is not given for free, but the followers must pay certain percentage ϵ of their gain to their respective local leaders. Again, the formula for wealth update is recursive, because the local leaders may be themselves followers and send part of the collected fee further to their local leaders, until the imitation chain ends. Introducing an auxiliary variable Y_i, and denoting $\Omega_i = \{j \in \Gamma_i : L(j) = i\}$ the set of agents who follow the agent i, the formula is

$$W_i(t+1) - W_i(t) = (1 - \epsilon\, l_i)\, Y_i(t)$$
$$Y_i(t) = \frac{1}{2}\Big[1 - a_i(t)\,\mathrm{sign}\Big(\sum_{j=1}^{N} a_j(t)\Big)\Big] + \epsilon \sum_{j\in\Omega_i} Y_j(t). \quad (3)$$

Clearly, $Y_i(t)$ is the profit acquired by the agent i at time t. The first term in the second line of (3) represents the direct gain depending on the action a_i of the agent i, as the minority rule dictates. The second term is the fee the agent i collected from her followers. From this profit, the agent sends $\epsilon\, l_i\, Y_i(t)$ to her local leader. Hence the expression in the first line of (3).

3 Imitation networks

3.1 Generally

First of all, we note that the imitation network is a collection of oriented trees; the edges go from the imitator to the imitated. In real world the situation may not be so simple, as we can imagine agents who compile pieces of information from two or more sources, combining them into a more complex strategy. And indeed, the empirical study of influence network [58] reveals a structure which is not a simple tree. All this is neglected in our model.

Each of the trees contains just one leader, which is the root. So, the tree is an imitation domain taking the same action. The first question we ask is the size distribution $P(d)$ of the imitation domains. The network evolves in time and we shall see that sometimes the evolution is rather slow. We shall observe snapshots of the distribution at several times and more detailed evolution of some of its features, like the number of domains n_{dom} and the size of the largest one d_{max}.

More detailed insights in the structure of imitation trees is provided by the degree distribution. Obviously, the out-degree is always 1, but the in-degree is non-trivial. In this work we shall call it forking and for the agent i it is $f_i = |\Omega_i|$. Besides the distribution of domain sizes, the distribution of forking $P(f)$ will be the most interesting quantity.

As for the substrate network, three types of graphs will be studied. First of them is the one-dimensional chain, or rather ring, because we use periodic boundary conditions. The edges connect only the nearest neighbours in this chain and the imitation network consists of one dimensional segments. The forking can assume only values $f = 0$, 1, or 2 and is therefore trivial.

The second type of substrate is the Erdös-Rényi (ER) random graph [49] where each pair of vertices is connected with the same edge probability q. Important feature of the ER graph is that the degree distribution is strongly peaked around the value qN, so that the graph is very homogeneous.

This is not the case for the third substrate investigated. It is the Barabási-Albert graph [57], constructed via a graph process, where placing of new edges is determined by the positions of already existing edges. At step i we connect the vertex i by m new edges with some of the vertices $1, 2, \ldots, i-1$. The probability that we connect it to the vertex j is proportional to $k_j + b$, where k_j is the number of edges already connected to j from vertices $>j$ and b is the second parameter of the model, besides the number m. The simplest choice is $b = m$ and we shall use it also here. The process generates a graph with very broad degree distribution, which asymptotically follows a power law. However, we must note that for the typical size used in this work, $N = 1001$, the BA graph is not yet in the asymptotic regime and it would be too daring to claim that the substrate has power-law degree distribution.

The fourth type of substrate is the Barabási-Albert 'model A' graph [57], which differs from the usual BA model by absence of preferential attachment, i.e. in the graph process the new m edges connect the vertex i with any of the vertices $1, \ldots, i-1$ with equal probability. In this case, the degree distribution is exponential.

3.2 Linear chain

In this case it is easy to visualise the imitation domains. We show in Figures 1 and 2 the typical distribution of wealth among agents, as it evolves in time. The leaders are located at local maxima of the curves, while the local minima mark the boundaries between different imitation domains. The evolution tends to coarsen the domains, as some of the domains are swallowed by other ones. We can also see that the overall character of the "landscapes" depend on the cost of information ϵ. For smaller ϵ the slope of the valleys is more or less uniform; for larger ϵ the bottom of the valleys is more flat, but in the vicinity of the leader the slope is much higher.

Let us now look at the evolution of the number of domains and size of the largest domain, as shown in Figure 3. We do not observe any noticeable dependence on the memory length M, but the influence of the information cost is significant. Although the general character of the time dependence remains the same, larger ϵ markedly slows down the evolution. But the most important observation is that in all cases investigated the final state

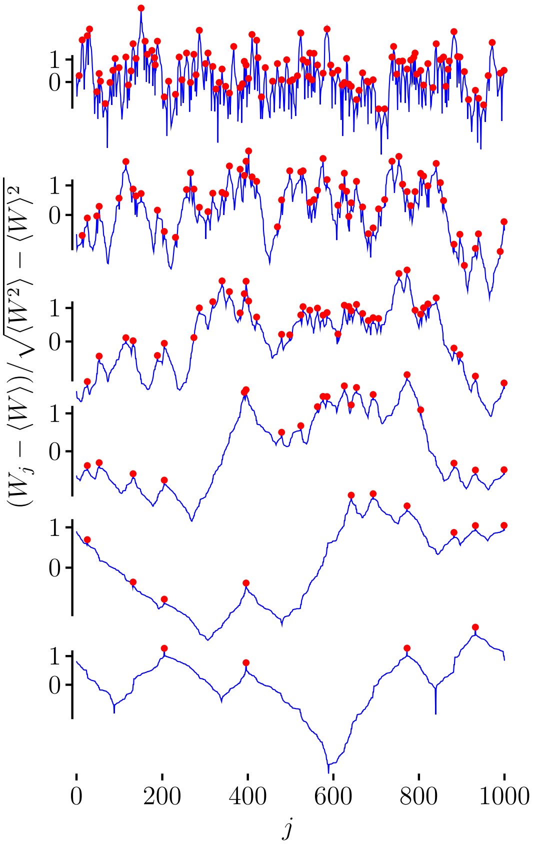

Fig. 1. Imitation on linear chain of length $N = 1001$. Snapshots of the distribution of wealth among agents at times (from tom to bottom) $t = 885$, 26 366, 61 584, 143 844, 335 981, and 784 759. The positions of the leaders are indicated by bullets. The memory length is $M = 6$; the information cost is $\epsilon = 0.003$.

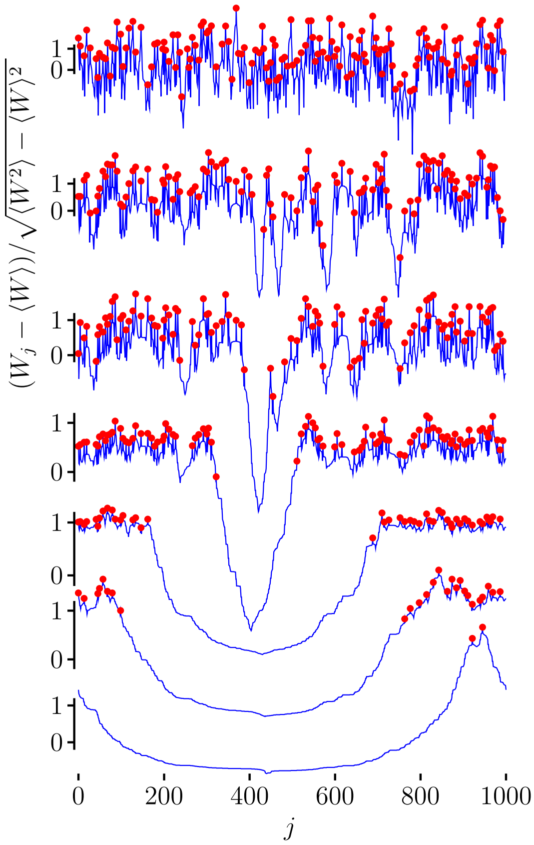

Fig. 2. The same as Figure 1 for $\epsilon = 0.01$. Times are (from tom to bottom) $t = 2335$, 112 883, 297 635, 784 759, 5 455 594, 14 384 498, and 10^8.

consists of two domains only, each of them comprising approximately half of the agents. The balance in the sizes of the remaining two domains can be understood as a feedback effect from the minority rule to the imitation network structure. Indeed, the domain which is larger must always loose. Only the smaller domain can sometimes win. But as it wins, it headhunts new adherents from the edge of the larger domain, until it becomes itself larger than the other domain and the growth reverts. An important lesson from Figure 3 is also the scale of time when the final state is reached. For example, if one ended the evolution at step 10^6 for $\epsilon = 0.01$, it would seem that we are close to a stationary state with several tens of domains. But this would be wrong, as we would have reached only a long-lived metastable or transient state. After 10^7 steps it relatively quickly drops to the true stable state with two domains.

This also means that the stationary domain size distribution is trivial. On the other hand, the transient distribution is not and we show its shape, as it evolves in time, in Figure 4. We found that typically in the transient regime there is one large domain and many small ones. Therefore, in Figure 4 we show the distribution of all domains except the largest one. Clearly, the distribution has exponential tail which becomes broader as time passes. This is the quantitative manifestation of the coarsening of domains we observed qualitatively in Figures 1 and 2. The same type of behaviour was seen for any value of the memory M and the information cost ϵ. The domains look as if the leaders were scattered randomly and covered the parts of the chain which go up to a point somewhere between themselves and the closest leader. Indeed, it is an elementary result from probability theory that distribution of distances between points distributed randomly on a line is exponential. This finding suggests that in the transient regime the feedback of the minority rule on the structure of the network is weak and becomes significant only in the true stationary state or close to it.

It is very instructive to look also at the evolution of the volatility. To this end, we define time-resolved average of square attendance, with exponentially decaying kernel

$$\langle A^2 \rangle_t = \lambda \left(\sum_{i=1}^{N} a_i(t) \right)^2 + (1-\lambda)\langle A^2 \rangle_{t-1}. \tag{4}$$

We use $\lambda = 0.01$, which implies effectively averaging over last 100 steps. The results shown in Figure 5 demonstrate strong dependence on the memory length. For lower M we are in the crowded phase and creation of domains substantially lowers the volatility Recall that lower volatility

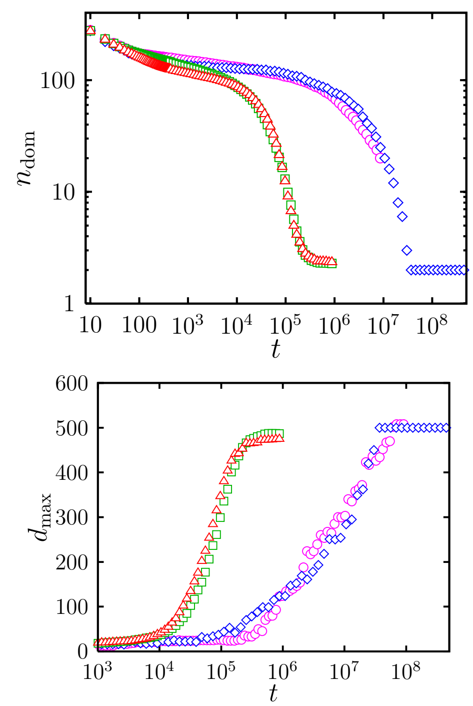

Fig. 3. Time evolution of the number of imitation domains (upper panel) and the size of the largest domain (lower panel), on a linear chain of length $N = 1001$. The memory length is $M = 6$ (○ and □), $M = 10$ (△ and ◇); the information cost is $\epsilon = 0.003$ (□ and △) and $\epsilon = 0.01$ (○ and ◇).

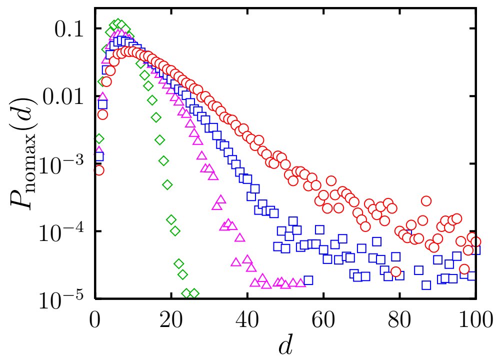

Fig. 4. Distribution of imitation domain sizes on a linear chain of length $N = 1001$. The largest domain is excluded from this statistics. Data take at times $t = 694$ (◇), $t = 6157$ (△), $t = 12\,741$ (□), and $t = 26\,365$ (○). The memory length is $M = 6$; the information cost is $\epsilon = 0.003$.

means more effective system, i.e. in this regime imitation means benefit for the system as a whole. But this holds only in the transient period. When the domains start growing beyond certain level, the volatility grows until it settles at a value which is orders of magnitude larger than the volatility in standard MG. The selfish desire of the agents to copy from more successful neighbours eventually leads to disaster for the whole ensemble. This is the imitation dilemma at work.

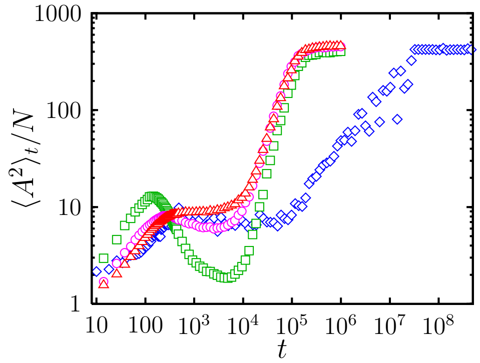

Fig. 5. Time evolution of the square attendance averaged with exponentially decaying kernel. Imitation occurs on a linear chain of length $N = 1001$. Memory and information cost are $\epsilon = 0.01$, $M = 10$ (◇) and $\epsilon = 0.003$, $M = 6$ (□), $M = 8$ (○), $M = 10$ (△).

We also perceive the influence of the information cost in Figure 5. Although the ultimate fate is the same, larger ϵ prolongs the transient period. A partial explanation may be found when comparing Figures 1 and 2. As we already mentioned, the peaks topped by leaders are sharper for larger ϵ, so it is less likely that one domain swallows the other.

3.3 Erdös-Rényi graph

Similar study was carried for the ER substrate network. The evolution of the distribution of domain sizes is shown in Figure 6. We clearly see the tendency towards formation of a single peak around the average domain size, although we were not able to make runs long enough to be sure about what is the stationary state. However, it seems that several domains (more than two) of approximately the same size are formed. There is an interesting difference between chain and ER substrates in the transient regime. While we have seen that the chain accommodates exponentially distributed domains, on ER graph the domain sizes are distributed according to the power law

$$P(d) \sim d^{-\delta} \qquad (5)$$

with $\delta \simeq 1.2$. This power-law distribution is formed almost instantly and then slowly breaks down when the peak is formed.

Surprisingly, the forking distribution remains almost unchanged in the time evolution, as testified in Figure 7, only the cutoff at highest f moves to higher values, as the domains grow. Obviously, f cannot exceed the domain size. This observation suggests that the evolution of the domains proceeds by merging smaller domains into larger ones, while the internal structure of the domains is kept essentially unchanged. In Figure 7 we can also see that neither the information cost has any influence on the forking distribution. We can conclude that the power law

$$P(f) \sim f^{-1} \qquad (6)$$

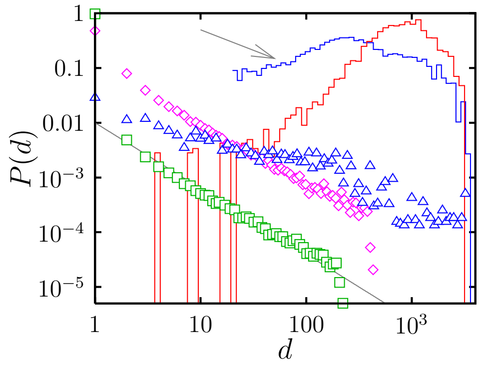

Fig. 6. Distribution of sizes of the imitation domains on Erdös-Rényi graph with $N = 4001$ and edge probability $q = 0.1$. The distribution is plotted by points at times $t = 2$ (□), $t = 6$ (◇), and $t = 11$ (△). The histograms show the distribution at times $t = 11$ (histogram pointed at by the arrow) and $t = 10\,000$. The memory length is $M = 7$; the information cost is $\epsilon = 0$. The straight line is the power-law dependence $\propto d^{-1.2}$.

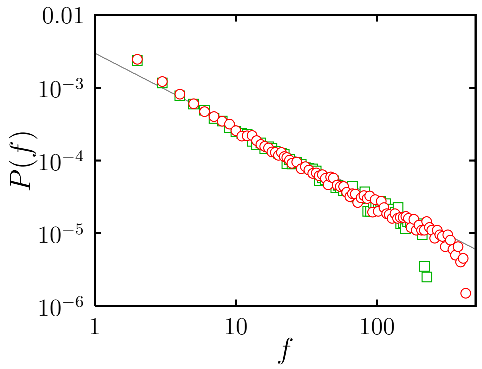

Fig. 7. Forking distribution on ER graph with $N = 4001$ and edge probability $q = 0.1$, at times $t = 6$ (□) and $t = 10\,000$ (○). The memory length is $M = 7$; the information cost is $\epsilon = 0$. The straight line is the power-law dependence $\propto f^{-1}$.

holds universally on ER graph. Exactly the same result was found in reference [48] and this is another argument on favour of the universality of the distribution (6), which seems to be insensitive to the details of the procedure of formation of the imitation network.

3.4 Barabási-Albert graph

We use BA graph with parameters $m = b = 2$. We have not seen much difference for different choices of the parameters.

Contrary to the ER graph, BA graphs are very inhomogeneous. The degree distribution is broad. We believe that it is not so much important whether it is strictly a power law or not. The essential point is that the tail of the distribution falls off slowly enough.

As with the one-dimensional chain, we first investigate temporal aspects of the domain distribution. In Figure 9 we can see the evolution of the number of domains and size of the largest domain. The most striking feature is

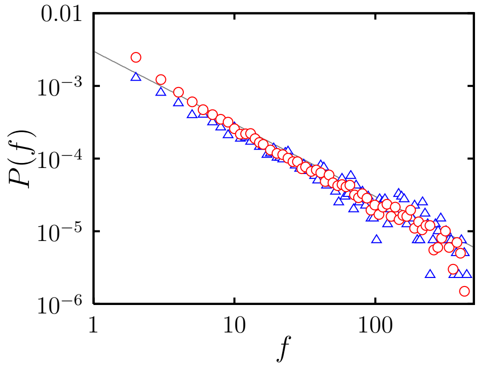

Fig. 8. Forking distribution on ER graph with $N = 4001$ and edge probability $q = 0.1$. The memory length is $M = 7$; the information cost is $\epsilon = 0$ (○) and $\epsilon = 0.1$ (△). The straight line is the power-law dependence $\propto f^{-1}$.

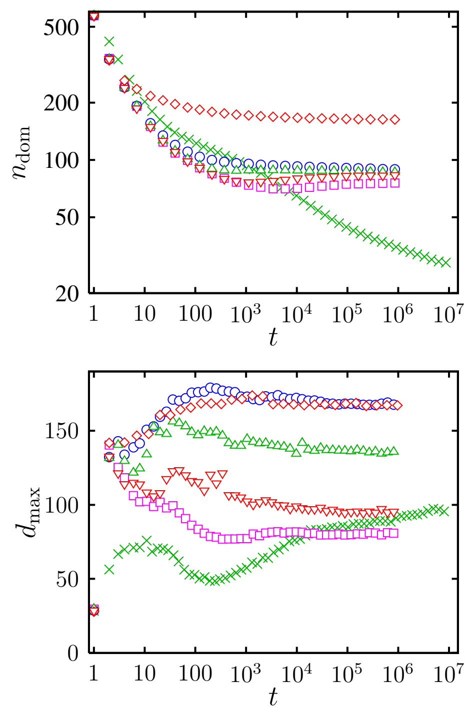

Fig. 9. Time evolution of the number of domains (upper panel), and size of the largest domain (lower panel) on Barabási-Albert graph with $N = 1001$ and parameters $m = b = 2$. The memory length is $M = 5$ and the cost of the information is $\epsilon = 0$, (×), 0.01 (□), 0.02 (▽), 0.05 (△), 0.1 (○), and 0.5 (◇).

different behaviour for zero and positive information cost. When $\epsilon = 0$, the evolution does not seem to end within the time we were technically able to simulate. The number of domains keeps decreasing and the largest domain keeps growing. On the other hand for $\epsilon > 0$ a stationary state seems to be reached. Of course, we cannot be sure that we reached a true stationarity. In fact, it is very probable that we see only a very long-lived transient state, as was the case in one-dimensional chain, and because higher ϵ hinders the approach to the true equilibrium, we are unable

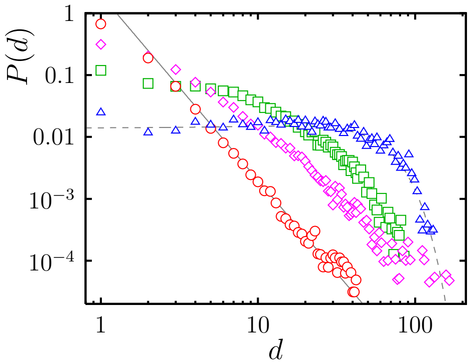

Fig. 10. Distribution of sizes of the imitation domains on Barabási-Albert graph with $N = 1001$ and parameters $m = b = 2$. The distribution is measured at times $t = 2$ (○), $t = 12$ (◇), $t = 2068$ (□), and $t = 10^7$ (△). The memory length is $M = 5$; the information cost is $\epsilon = 0$. The straight line is the power-law dependence $\propto d^{-3}$, the dashed line is the Gaussian $\propto \exp\left(-\left[(d - d_0)/d_1\right]^2\right)$ with $d_0 = 15$ and $d_1 = 55$.

to observe it. For simplicity we shall speak of stationary state even though it may be in fact a transient.

The parameters of the stationary state depend on ϵ in a subtle way. The number of domains increases very weakly when we go $\epsilon = 0.01$ to 0.1, while the largest domain increases too, and rather steeply. This can mean only one thing: the distribution broadens when we increase ϵ. In Figures 10 and 11 we observe snapshots of the domain size distribution at several times, for $\epsilon = 0$, 0.01, and 0.1. We can see that for $\epsilon = 0$ the domain distribution is power-law initially, with exponent close to $\delta \simeq 3$ and then gradually develops a peak, which we fitted relatively well on a Gaussian. The situation we found in the one-dimensional chain and ER graph repeats here. The domains balance their sizes so that none of them dominates. This is clearly due to the feedback of the minority mechanism on the imitation network structure. For slightly larger information cost $\epsilon = 0.01$ the situation starts to change. Again, at early stages the distribution is power-law with $\delta \simeq 3$ but then the exponent decreases and eventually the tail ceases to be power-law, although the peak is not formed either. For still larger value $\epsilon = 0.1$ the stationary state is characterised by power-law tail with exponent $\delta \simeq 2$.

The overall scenario seems to be the following. Very quickly, power-law distributed domains are formed, with exponent $\delta \simeq 3$. Then, the domains coarsen by merging and the exponent drops to the value $\delta \simeq 2$. But when some domains become so large that the minority rule disfavours their members, their growth stops or they may even shrink, as some agents within the imitation tree split off and skip to another domain. Such process dissolves the power-law tail and eventually leads to a distribution peaked around the average domain size. Now comes the influence of the information cost. We have seen that larger ϵ slows down the process of merging the domains. The stationary state is reached, which is the less influenced by the minority mechanism the larger the information cost is. If we consider the steady evolution with $\epsilon = 0$ as a reference

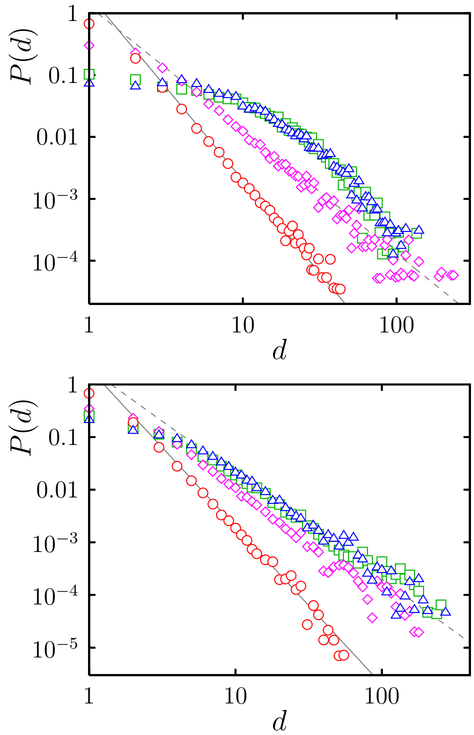

Fig. 11. Distribution of sizes of the imitation domains on BA graph with $N = 1001$ and parameters $m = b = 2$. The distribution is measured at times $t = 2$ (○), $t = 8$ (◇), $t = 695$ (□), and $t = 10^6$ (△). The memory length is $M = 5$; the information cost is $\epsilon = 0.01$ (upper panel) and $\epsilon = 0.1$ (lower panel). The solid lines are the power-law dependence $\propto d^{-3}$, while the dashed line denotes the power law $\propto d^{-2}$.

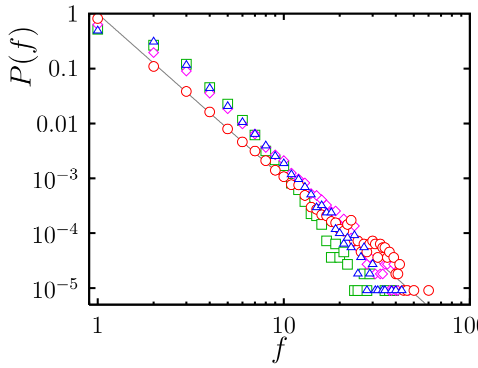

Fig. 12. Distribution of forking on Barabási-Albert graph with $N = 1001$ and parameters $m = b = 2$. The distribution is measured at times $t = 2$ (○), $t = 12$ (◇), $t = 2068$ (□), and $t = 10^7$ (△). The memory length is $M = 5$; the information cost is $\epsilon = 0$. The straight line is the power-law dependence $\propto f^{-3}$.

behaviour, positive ϵ stops the evolution at certain stage, the earlier the greater ϵ is. For example for $\epsilon = 0.01$ the power-law tail breaks down, but the peak does not have time to develop. For $\epsilon = 0.1$ the power-law tail does not have time to vanish and remains there.

The forking behaves differently than it did on ER graph. We show in Figures 12 and 13 how the forking distribution evolves in time. The change is not spectacular

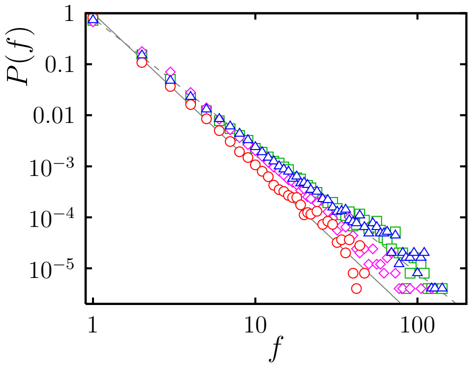

Fig. 13. Distribution of forking on Barabási-Albert graph with $N = 1001$ and parameters $m = b = 2$. The distribution is measured at times $t = 2$ (○), $t = 8$ (◇), $t = 695$ (□), and $t = 10^6$ (△). The memory length is $M = 5$; the information cost is $\epsilon = 0.1$. The solid line is the power-law dependence $\propto f^{-3}$, while the dashed line denotes the power law $\propto f^{-2.5}$.

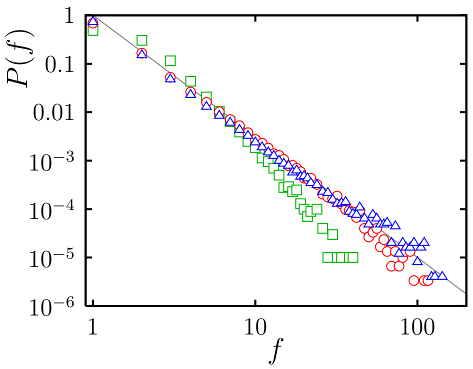

Fig. 14. Distribution of forking on Barabási-Albert graph with $N = 1001$ and parameters $m = b = 2$. The memory length is $M = 5$. The information cost is $\epsilon = 0$ (□), 0.01 (○), and 0.1 (△). The line is the power-law dependence $\propto f^{-2.5}$.

but cannot be overlooked. Here also the information cost makes big difference. The distribution which emerges quasi instantly is power law $P(f) \sim f^{-\phi}$ with $\phi \simeq 3$. But for $\epsilon = 0$ the power-law tail is gradually lost and large forkings are suppressed, while for $\epsilon = 0.1$ large forkings are enhanced instead, and power law with lower exponent $\phi \simeq 2.5$ results. We can compare the ultimate distributions for $\epsilon = 0$, 0.01, and 0.1 in Figure 14. Power-law dependence is clearly seen for non-zero information cost, while the distribution for $\epsilon = 0$ decays clearly faster than a power law. Therefore, contrary to the ER graph the evolution implies not only merging domains but also change in their internal structure. Details of this process are not clear yet.

We already stressed several times that BA graph is very inhomogeneous. Agents who sit on vertices with high substrate degree may behave differently than those who have only few links on the substrate graph. To see better the difference, we investigated average forking and average wealth conditioned to the degree of the vertex on the

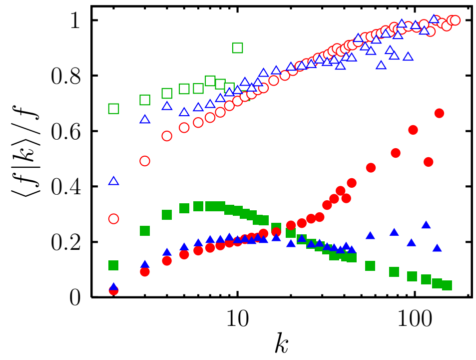

Fig. 15. Average forking conditional to the degree of the vertex on the substrate graph, for Barabási-Albert graph with $N = 1001$ and parameters $m = b = 2$. The empty and filled symbols correspond to leaders and imitators, respectively. The information cost is $\epsilon = 0$ (□ and ■), 0.01 (△ and ▲), 0.1 (○ and ●).

substrate graph. For the forking we define

$$\langle f|k\rangle = \frac{\sum_{i=1}^{N} |\Omega_i| \delta(|\Gamma_i| - k)}{\sum_{i=1}^{N} \delta(|\Gamma_i| - k)} \qquad (7)$$

where by $\delta(x)$ we denote the Kronecker delta. As for the wealth, it is convenient to use the reduced value, which, for of agent i, is

$$\widetilde{W}_i = \frac{N W_i}{\sum_{j=1}^{N} W_j} - 1. \qquad (8)$$

The conditional average is then, in analogy with (7),

$$\langle \widetilde{W}|k\rangle = \frac{\sum_{i=1}^{N} \widetilde{W}_i \delta(|\Gamma_i| - k)}{\sum_{i=1}^{N} \delta(|\Gamma_i| - k)}. \qquad (9)$$

We shall also distinguish between leaders and imitators. The data are shown in Figures 15 and 16. A surprising result is observed for conditional average forking of the leaders, for $\langle f|k\rangle$ grows faster than k for any ϵ, although for larger ϵ the effect is more pronounced. Followers behave differently. Zero information cost implies slower increase than k, intermediate value leads to proportionality with k and high values of ϵ cause faster-than-linear growth also for the followers. A possible explanation is the following. When imitation becomes expensive, agents lying high on the imitation tree collect more wealth and therefore can attract more followers. But the number of followers is limited by the degree k of the vertex on the substrate. If the buildup of the imitation tree was random, the number of followers a local leader gets would be proportional to k. But this would induce larger wealth collected as imitation fee, and therefore the probability to get further followers is enhanced. Proportion of substrate edges used in the imitation tree is therefore higher for vertices with higher substrate degree.

The conditional wealth shown in Figure 16 complements this picture. The first obvious observation is, that

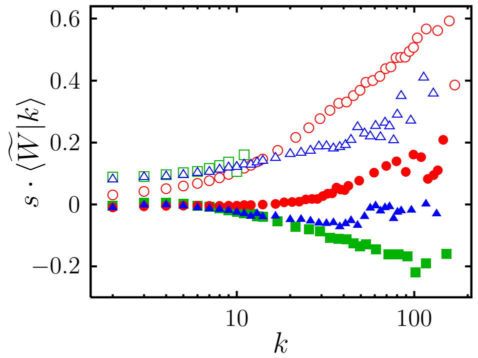

Fig. 16. Average reduced wealth $\widetilde{W} = W/\langle W \rangle - 1$ conditional to the degree of the vertex on the substrate graph, for Barabási-Albert graph with $N = 1001$ and parameters $m = b = 2$. The empty and filled symbols correspond to leaders and imitators, respectively. The information cost is $\epsilon = 0$ (□ and ■), 0.01 (△ and ▲), 0.1 (○ and ●). To accommodate all data in one figure, the auxiliary factor s is introduced, with $s = 10$, 1, 0.1 for $\epsilon = 0$, 0.01, and 0.1, respectively.

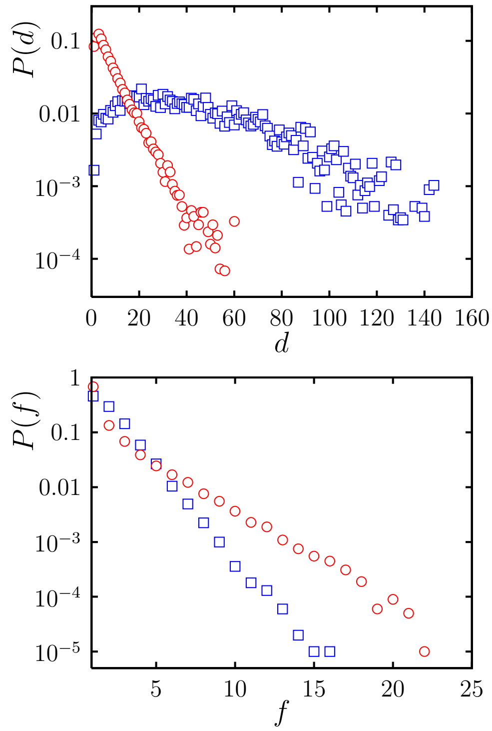

Fig. 17. Distribution of imitation domain sizes (upper panel) and forking (lower panel) on Barabási-Albert 'model A' graph, with $N = 1001$ and $m = 2$. The memory length is $M = 5$. The information cost is $\epsilon = 0$ (□) and 0.1 (○).

the leaders have wealth consistently above average and the followers below. Only followers with high substrate vertex degree rise beyond average, and they are few. For $\epsilon > 0$ leaders with higher k have always larger wealth. On the other hand, the wealth of the followers grows with k only for large enough ϵ and k. For zero information cost the wealth of followers decreases with k instead.

3.5 Barabási-Albert 'model A' graph

In this case, the substrate degree distribution is exponential and the properties of the imitation network change accordingly, compared to the BA graph investigated in the previous section. However, some features remain in force. For example the domain size distribution plotted in Figure 17 shows that for zero information cost the distribution is peaked, while for $\epsilon = 0.1$ it is not, but has exponential shape like in the transient state of the one-dimensional chain. The forking in Figure 17 is distributed exponentially for all values of ϵ investigated, but larger value of the information cost results in broader distribution. We can conclude that for emergence of power-law distributed forking it is necessary that there are enough vertices with large degree in the substrate. This is satisfied in ER and BA graph, but not in linear chain or BA 'model A' graph.

4 Conclusions

Minority Game proved to be a very productive framework for studying the effect of social imitation. The four substrate graphs studied, i.e. linear chain, ER graph, BA graph and BA 'model A' graph, share some general features but show significant differences in others. We looked at the structure of imitation domains. By definition, they are oriented trees, which at the same time must be subgraphs of the substrate. This imposes limits on their structure; e.g. on the one-dimensional chain their topology is trivial and the only interesting parameter is their length.

The domains are dynamic objects. They slowly grow, merge, and exchange agents among each other. The minority character of the game discourages the agents to join a domain which is too large, because this enhances the chance to be in the majority. We observed clear tendency to form a few equally sized domains. For the chain substrate we have shown that the final state consists of only two domains comprising nearly exactly half of the agents each. For other substrates, the runs were not long enough to be sure about the final outcome, but we consider very likely that the evolution would lead to the same result.

The important parameter was the information cost ϵ the imitators must pay to the imitated. Non-zero ϵ markedly slows down the approach to the stationary state conjectured above. Instead, long-living metastable, or transient, state develops, with highly non-trivial structure. Taking the case $\epsilon = 0$ as a reference, the transient state for higher ϵ looks as if the reference evolution was stopped at earlier time. Different values of ϵ are roughly equivalent to snapshots of the $\epsilon = 0$ evolution at different times.

It was the transient state we devoted most attention to. We found that the distribution of domain sizes is exponential if the substrate is either the one-dimensional chain or BA 'model A' graph. For ER and BA graphs power-law domain size distribution is formed almost instantly but then evolves towards a well-defined peak. For BA graph we first observe mere lowering of the exponent of the power-law distribution, then the tail is rounded by a cutoff and finally a peak develops. Depending on the value of ϵ the evolution of the distribution stops at different moments, the earlier the higher ϵ is.

The internal structure of the imitation domains was measured through the in-degree, which we call forking, of the imitation trees. For one-dimensional substrate it is trivial. For ER graph, we find power-law distribution with exponent $\phi \simeq 1$, in agreement with previous studies [48,56]. The exponent seems to be very robust and does not change even when the distribution of domain sizes changes dramatically. A possible explanation, suggested already by some conjectures raised in [48] is, that the imitation trees formed at the earliest stages of the evolution have barely random nature, with no reference to the rules of the Minority Game and/or the details of the model. Hence the insensitivity to the parameter ϵ and the memory length M and also to the differences between our model and the model of references [48,56]. Moreover, the forking distribution does not change in time, so the structure imprinted at the beginning seems to remain in force forever. Most probably the reason is that the imitation domains evolve by merging, without other structural changes.

The situation is somewhat different on the BA substrate graph. Also here a power-law forking distribution is created immediately, with exponent $\phi \simeq 3$, but then it changes, breaking the power-law at for $\epsilon = 0$, while for $\epsilon > 0$ the distribution tends to power law again, but with lower exponent $\phi \simeq 2.5$. This implies that on BA substrate the domains evolve not by mere aggregation, but also by internal rearrangement.

For BA 'model A' substrate the domain distribution has again exponential tail, as in the case of linear chain. This is not so surprising, as the degree distribution in BA 'model A' is exponential and therefore large degrees are very unlikely, while most probable degree is about 3 (for the value $m = 2$ used in our simulation) independently of the system size. So, the substrate is very close to a regular tree of degree 3, while linear chain is in fact regular a three of degree 2. Note the difference from ER graph, where the tail decays faster than exponentially, but the peak is located at large value, proportional to the system size N, if the edge probability q is kept fixed.

The forking distribution for BA 'model A' is exponential, which is dictated by the exponential degree distribution of the substrate. Indeed, there are few nodes with high substrate degree, so there is no chance to find many agents with large forking.

For BA substrate we also investigated the correlations between forking and wealth on one side and substrate degree on the other. We found that the leaders and followers behave differently and there is also strong dependence on the information cost ϵ. Most importantly, the forking at the leaders grows faster than linearly with the substrate degree and this effect is the stronger the larger ϵ is. Agents

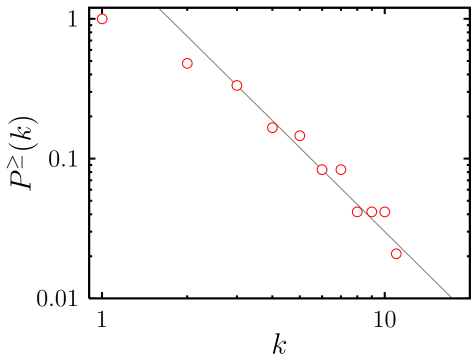

Fig. 18. Cumulative degree distribution in the network of influence. The data were extracted from Figure 4 of reference [58]. The straight line is the power law $\propto k^{-2}$.

with higher degree are more effective in using their edges to attract followers. For the forking at the followers, the same effect appears only for large enough ϵ. For zero information cost, the dependence on the substrate degree is sub-linear. The followers with larger degree attract less other followers than they would if the edges were exploited randomly.

This feature is certainly due to the fact that an agent with more neighbours can collect more wealth as a fee from her imitators. But higher wealth implies even more imitators. Hence the super-linear dependence of the forking on substrate degree. This picture is supported by the dependence of average wealth on substrate degree, which is growing function for the leaders. On the other hand, it is decreasing for the followers if the information cost is zero. Only for large enough ϵ the wealth of followers is also growing function of substrate degree.

Finally, to compare our result with reality, we extracted the degree distribution from the network of mutual influence published in Figure 4 of reference [58]. We show the extracted data in our Figure 18. The first impression is that the network is too small and the distribution covers too narrow range to make any firm conclusion. However, the data at least do not exclude the possibility of power-law degree distribution with exponent close to 2. In our simulations the exponent was either significantly smaller, for ER substrate, or somewhat larger, for BA substrate. So, qualitatively the model seems to be compatible with reality, but there is quantitative disagreement in the exponent. The comparison is further complicated by the fact that the influence network of reference [58] is not a tree but a more complicated structure. Clearly, comparison with further and more extensive empirical data is necessary if we want to assess practical relevance of our imitation models.

This work was supported by the MŠMT of the Czech Republic, grant No. 1P04OCP10.001, and by the Research Program CTS MSM 0021620845.

References

1. W.B. Arthur, Amer. Econ. Rev. (Papers and Proceedings) **84**, 406 (1994)
2. D. Challet, Y.-C. Zhang, Physica A **246**, 407 (1997)
3. R. Savit, R. Manuca, R. Riolo, e-print `arXiv:adap-org/9712006`
4. R. Savit, R. Manuca, R. Riolo, Phys. Rev. Lett. **82**, 2203 (1999)
5. D. Challet, Y.-C. Zhang, Physica A **256**, 514 (1998)
6. D. Challet, M. Marsili, Phys. Rev. E **60**, R6271 (1999)
7. N.F. Johnson, S. Jarvis, R. Jonson, P. Cheung, Y.R. Kwong, P.M. Hui, Physica A **258**, 230 (1998)
8. N.F. Johnson, P.M. Hui, R. Jonson, T.S. Lo, Phys. Rev. Lett. **82**, 3360 (1999)
9. N.F. Johnson, D.J.T. Leonard, P.M. Hui, T.S. Lo, Physica A **283**, 568 (2000)
10. R. D'Hulst, G.J. Rodgers, M. Ausloos, e-print `arXiv:adap-org/9907003`
11. A. Cavagna, J.P. Garrahan, I. Giardina, D. Sherrington, Phys. Rev. Lett. **83**, 4429 (1999)
12. M.A.R. de Cara, O. Pla, F. Guinea, Eur. Phys. J. B **13**, 413 (2000)
13. J.P. Garrahan, E. Moro, D. Sherrington, Phys. Rev. E **62**, R9 (2000)
14. J.P. Garrahan, E. Moro, D. Sherrington, e-print `arXiv:cond-mat/0012269`
15. A. Cavagna, Phys. Rev. E **59**, R3783 (1999)
16. D. Challet, M. Marsili, R. Zecchina, Phys. Rev. Lett. **84**, 1824 (2000)
17. M. Marsili, D. Challet, R. Zecchina, Physica A **280**, 522 (2000)
18. D. Challet, M. Marsili, R. Zecchina, Int. J. Theor. Appl. Finance **3**, 451 (2000)
19. M. Marsili, D. Challet, Phys. Rev. E **64**, 056138 (2001)
20. T.S. Lo, P.M. Hui, N.F. Johnson, Phys. Rev. E **62**, 4393 (2000)
21. M. Hart, P. Jefferies, P.M. Hui, N.F. Johnson, Eur. Phys. J. B **20**, 547 (2001)
22. J.A.F. Heimel, A.C.C. Coolen, Phys. Rev. E **63**, 056121 (2001)
23. A.C.C. Coolen, J.A.F. Heimel, D. Sherrington, Phys. Rev. E **65**, 016126 (2002)
24. J.A.F. Heimel, A. De Martino, J. Phys. A: Math. Gen. **34**, L539 (2001)
25. T. Galla, A.C.C. Coolen, D. Sherrington, J. Phys. A: Math. Gen. **36**, 11159 (2003)
26. A.C.C. Coolen, *The Mathematical Theory of Minority Games* (Oxford University Press, Oxford, 2005)
27. D. Challet, M. Marsili, Y.-C. Zhang, *Minority Games* (Oxford University Press, Oxford, 2005)
28. N.F. Johnson, P. Jefferies, P.M. Hui, *Financial Market Complexity* (Oxford University Press, Oxford, 2003)
29. Y.-C. Zhang, Europhys. News **29**, 51 (1998)
30. D. Challet, A. Chessa, M. Marsili, Y.-C. Zhang, Quantitative Finance **1**, 168 (2001)
31. D. Challet, M. Marsili, Y.-C. Zhang, Physica A **276**, 284 (2000)
32. P. Jefferies, M. Hart, P.M. Hui, N.F. Johnson, Eur. Phys. J. B **20**, 493 (2001)
33. D. Challet, M. Marsili, Y.-C. Zhang, Physica A **294**, 514 (2001)
34. I. Giardina, J.-P. Bouchaud, M. Mézard, e-print `arXiv:cond-mat/0105076`
35. S. Moelbert, P. De Los Rios, Physica A **303**, 217 (2002)
36. T. Kalinowski, H.-J. Schulz, M. Briese, Physica A **277**, 502 (2000)
37. F. Schweitzer, J. Zimmermann, H. Mühlenbein, Physica A **303**, 189 (2002)
38. E. Burgos, H. Ceva, R.P.J. Perazzo, Physica A **337**, 635 (2004)
39. E. Burgos, H. Ceva, R.P.J. Perazzo, Physica A **354**, 518 (2005)
40. G. Fagiolo, M. Valente, Computational Economics **25**, 41 (2005)
41. F. Slanina, Physica A **286**, 367 (2000)
42. F. Slanina, Physica A **299**, 334 (2001)
43. H.-J. Quan, B.-H. Wang, P.M. Hui, X.-S. Luo, Chinese Phys. Lett. **18**, 1156 (2001)
44. H. Quan, B.-H. Wang, P.M. Hui, Physica A **312**, 619 (2002)
45. H.-J. Quan, B.-H. Wang, W.-S. Yang, W.-N. Wang, X.-S. Luo, Acta Physica Sinica **51**, (2670)
46. H.-J. Quan, B.-H. Wang, P.M. Hui, X.-S. Luo, Physica A **321**, 300 (2003)
47. L. Shang, X.F. Wang, Physica A **361**, 643 (2006)
48. M. Anghel, Z. Toroczkai, K.E. Bassler, G. Korniss, Phys. Rev. Lett. **92**, 058701 (2004)
49. B. Bollobás, *Random Graphs* (Academic Press, London, 1985)
50. S.C. Choe, N.F. Johnson, P.M. Hui, Phys. Rev. E **70**, 055101 (2004)
51. S. Gourley, S.C. Choe, P.M. Hui, N.F. Johnson, e-print `arXiv:cond-mat/0401527`
52. S. Gourley, S.C. Choe, P.M. Hui, N.F. Johnson, Europhys. Lett. **67**, 867 (2004)
53. T.S. Lo, H.Y. Chan, P.M. Hui, N.F. Johnson, Phys. Rev. E **70**, 05610 (2004)
54. T.S. Lo, H.Y. Chan, P.M. Hui, N.F. Johnson, Phys. Rev. E **71**, 050101 (2005)
55. S.H. Lee, H. Jeong, J. Kor. Phys. Soc. **48**, S186 (2006)
56. S.H. Lee, H. Jeong, Phys. Rev. E **74**, 026118 (2006)
57. A.-L. Barabási, R. Albert, Science **286**, 509 (1999)
58. L. Kullmann, J. Kertész, K. Kaski, Phys. Rev. E **66**, 026125 (2002)